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INVARIANCE PROPERTIES OF RANDOM VECTORS AND STOCHASTIC PROCESSES BASED ON THE ZONOID CONCEPT

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Abstract

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Keywords: invariance; zonoid; exchangeability; ergodic theorem; isometry

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Invariance properties of random vectors and stochastic processes based on the zonoid concept*

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Abstract

Two integrable random vectors ξ and ξ^* in $\mathbb{R}^d$ are said to be zonoid equivalent if, for each $u \in \mathbb{R}^d$, the scalar products $\langle \xi, u\rangle$ and $\langle \xi^*, u\rangle$ have the same first absolute moments. The paper analyses stochastic processes whose finite-dimensional distributions are zonoid equivalent with respect to time shift (zonoid stationarity) and permutation of time moments (swap-invariance). While the first concept is weaker than the stationarity, the second one is a weakening of the exchangeability property. It is shown that nonetheless the ergodic theorem holds for swap invariant sequences and the limits are characterised.

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1 Introduction

Zonoids form an important family of convex bodies (i.e. convex compact sets) in the Euclidean space $\mathbb{R}^d$, see [30]. *Zonoids* are obtained as limits of zonotopes in the Hausdorff metric, where zonotopes are Minkowski (elementwise) sums of a finite number of segments.

The sums of segments and the limits of sums can be interpreted as expectations of random segments. By translation, it is possible to assume that all segments are centred and so are of the form $[-\xi, \xi]$ for a random vector $\xi \in \mathbb{R}^d$. Recall that the *support function* of a set K in $\mathbb{R}^d$ is given by

$$h_K(u) = \sup\{\langle u, x\rangle :\ x \in K\}\,, \quad u \in \mathbb{R}^d\,,$$

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where $\langle u, x\rangle$ denotes the scalar product. The expectation of $[-\xi, \xi]$ is the convex set Z_ξ^o identified by its support function, which is equal to the expected support function of the segment (see [18, Sec. 2.1]), i.e.

$$h_{Z_\xi^o}(u) = \mathbf{E}|\langle u, \xi\rangle|, \qquad u \in \mathbb{R}^d .$$

If ξ is integrable, Z_ξ^o is a centrally symmetric convex body (compact convex set).

A version of zonoid associated with a random vector was considered by Koshevoy and Mosler, see [17] and [24]. Namely, the zonoid Z_ξ of ξ is the expectation of $[0, \xi]$ and so the support function of Z_ξ is given by

$$h_{Z_\xi}(u) = \mathbf{E}\langle u, \xi\rangle_+ , \qquad u \in \mathbb{R}^d ,$$

where $x_+ = \max(x, 0)$. In order to stress the difference between the two variants of zonoids we call Z_ξ^o the *centred zonoid* of ξ, see Section 6 for the comparison of the two concepts. Note that Z_ξ is also well defined for possibly non-integrable ξ, in which case Z_ξ is unbounded. Nonetheless from now on we always assume that all mentioned random variables and random vectors are absolutely integrable and not identically zeros.

The concept of zonoid is useful in multivariate statistics to define trimming and data depth, see [4, 24]. It is well known that the zonoid of ξ does not uniquely characterise its distribution. For instance, on the line, Z_ξ is the segment with end-points determined by the expectations of the positive and negative parts of ξ and Z_ξ^o is the segment with end-points $\pm\mathbf{E}|\xi|$.

Definition 1.1. Two random vectors ξ and ξ^* are called *zonoid equivalent* if their centred zonoids coincide, i.e. $Z_\xi^o = Z_{\xi^*}^o$.

Two random variables are zonoid equivalent if their absolute values have the same expectation. The zonoid equivalence of ξ and ξ^* means

$$\mathbf{E}|\langle u, \xi\rangle| = \mathbf{E}|\langle u, \xi^*\rangle| \tag{1.1}$$

for all $u \in \mathbb{R}$. Note that the zonoid equivalence is defined using the centred zonoids. It is possible to define the alternative concept of zonoid equivalence by requiring $Z_\xi = Z_{\xi^*}$, which is stronger than the condition imposed in Definition 1.1. In all places where non-centred zonoids appear, we explicitly mention zonoids of random vectors. If random vectors share the same expectation, e.g. if they have symmetric distributions, then the two equivalence concepts are identical, see Section 6.

Zonoid equivalence of random vectors obtained by permuting two of their coordinates has been investigated in [22] and for all possible permutations in [23] in view of financial applications.

It is possible to modify the definition of zonoid to ensure the uniqueness of the underlying random vector. For this, lift ξ into the space $\mathbb{R}^{d+1}$ by adding to it the additional first coordinate being identically one. The zonoid of the obtained vector $(1, \xi)$ is a convex set $\hat{Z}_\xi$ in $\mathbb{R}^{d+1}$ called the *lift zonoid* of ξ, so that the support function of the lift zonoid is

$$h_{\hat{Z}_\xi}(k, u) = \mathbf{E}(\langle u, \xi\rangle + k)_+ , \qquad u \in \mathbb{R}^d,\ k \in \mathbb{R} . \tag{1.2}$$

In finance, the right-hand side of (1.2) is the price of a basket call option with strike $-k$ for $k \leq 0$ (if the expectation is taken with respect to a chosen martingale measure). It is known that the lift zonoid uniquely determines the distribution of ξ, see [24, Th. 2.21]. The well-known result of Breeden and Litzenberger [2] saying that the prices of all call options determine the distribution of ξ now becomes a corollary of a general result for lift zonoids in case $d = 1$ and also can be easily rephrased for multidimensional ξ, see [21]. By homogeneity, it is possible to set $k = 1$.

The uniqueness results hold also for the centred variant of the lift zonoid obtained by replacing the positive part in the right-hand side of (1.2) with the absolute value. A result of Hardin [12, Th. 1.1] implies that the distribution of an integrable random vector ξ is uniquely determined by $\mathbf{E}|1 + \langle u, \xi\rangle|$ for all u, equivalently by the centred zonoid of $(1, \xi)$. In the following we show that, if ξ is symmetric, it is possible to replace 1 by any random variable taking values ± 1.

The paper starts with the analysis of the main implication of the zonoid equivalence. Namely, in Section 2 we show that the zonoid equivalence yields the equality of the expected values for each even one-homogeneous function of the random vectors. Section 3 emphasises relationships between the zonoid equivalence and isometries of subspaces of L^1.

Stochastic processes whose finite-dimensional distributions remain zonoid equivalent for time shifts are discussed in Section 4. This property is brought in relationship to the stationarity of related stable and max-stable processes through their LePage representations.

Section 5 introduces the swap-invariance property for a random sequence that amounts to the zonoid equivalence of each permutation of all its finite subsequences, which is a weaker version of the exchangeability property. We prove the ergodic theorem for swap-invariant sequences and characterise the limits, thereby generalising the classical results for exchangeable sequences.

Section 6 discusses relationships between centred and non-centred zonoids. Finally, Section 7 collects a number of relevant results concerning zonoids of particular distributions. It is shown that zonoids identify uniquely distributions from location-scale families under rather mild conditions. The special case of random vectors with positive coordinates is also analysed, in particular log-infinitely divisible laws being important in financial applications.

The consideration of (non-centred) zonoids makes it possible to study possibly non-integrable random vectors, which is left for a future work. The same relates to L^p-zonoids considered in [20]. A number of results of this paper can be generalised for random elements in Banach spaces along the lines of [1].

2 Expectations of homogeneous functions

Let $\mathcal{H}$ (respectively $\mathcal{H}_e$) denote the family of all (respectively even) measurable homogeneous functions $\mathbb{R}^d \mapsto \mathbb{R}_+$, so that $f(cx) = cf(x)$ for all $x \in \mathbb{R}^d$ and $c \geq 0$. A simple example of a function from $\mathcal{H}_e$ is the Euclidean norm $\|x\|$, $x \in \mathbb{R}^d$.

Proposition 2.1. *If ξ and ξ^* are zonoid equivalent, then* $\mathbf{E}\|\xi\| = \mathbf{E}\|\xi^*\|$.

Proof. The integral of the support function of a convex body K over the unit sphere is $\frac{1}{2}d\kappa_d b(K)$, where $b(K)$ is called the mean width of K and κ_d is the volume of the unit ball in $\mathbb{R}^d$. By changing the order of integral and expectation, it is easy to see that the mean width of Z^o_ξ equals the expected mean width of the segment $[-\xi,\xi]$. The mean width of this segment can be found from the Steiner formula [30, Eq. (4.1.1)], see also [30, p. 210], as $b([-\xi,\xi]) = 4\|\xi\|\kappa_{d-1}/(d\kappa_d)$. Thus, $\mathbf{E}\|\xi\| = b(Z^o_\xi)d\kappa_d/(4\kappa_{d-1})$ is uniquely determined by Z^o_ξ. □

Proposition 2.1 can be extended to general functions from $\mathcal{H}_e$.

Theorem 2.2. *Two random vectors ξ and ξ^* are zonoid equivalent if and only if $\mathbf{E}f(\xi) = \mathbf{E}f(\xi^*)$ for all $f \in \mathcal{H}_e$.*

Proof. Sufficiency is immediate, since $f(x) = |\langle u, x\rangle|$ belongs to $\mathcal{H}_e$.

Necessity. By Proposition 2.1, $\mathbf{E}\|\xi\| = \mathbf{E}\|\xi^*\| = c$. Define probability measure $\mathbf{Q}$ with density

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \frac{\|\xi\|}{c}$$

and another measure $\mathbf{Q}^*$ generated by ξ^* in the same way. Denote by $\mathbf{E_Q}$ the expectation with respect to $\mathbf{Q}$ (and respectively with respect to $\mathbf{Q}^*$). Then for all $u \in \mathbb{R}^d$

$$\frac{1}{c}\,\mathbf{E}|\langle u,\xi\rangle| = \frac{1}{c}\,\mathbf{E}|\langle u,\xi\rangle|\,\mathbb{1}_{\{\|\xi\|\neq 0\}} = \mathbf{E_Q}|\langle u, \frac{\xi}{\|\xi\|}\rangle|\,\mathbb{1}_{\{\|\xi\|\neq 0\}} = \mathbf{E_Q}|\langle u, \frac{\xi}{\|\xi\|}\rangle|\,,$$
$$\frac{1}{c}\,\mathbf{E}|\langle u,\xi^*\rangle| = \frac{1}{c}\,\mathbf{E}|\langle u,\xi^*\rangle|\,\mathbb{1}_{\{\|\xi^*\|\neq 0\}} = \mathbf{E_{Q^*}}|\langle u, \frac{\xi^*}{\|\xi^*\|}\rangle|\,\mathbb{1}_{\{\|\xi^*\|\neq 0\}} = \mathbf{E_{Q^*}}|\langle u, \frac{\xi^*}{\|\xi^*\|}\rangle|\,.$$

Therefore, $\xi/\|\xi\|$ under $\mathbf{Q}$ and $\xi^*/\|\xi^*\|$ under $\mathbf{Q}^*$ share the same zonoid. Define measure μ on the unit Euclidean sphere by setting $\mu(A) = \mathbf{Q}(\xi/\|\xi\| \in A)$ and correspondingly μ^*. The convex body Z^o_μ with the support function

$$h_{Z^o_\mu}(u) = \int_{\mathbb{S}^{d-1}} |\langle u,x\rangle|\mu(dx) = \mathbf{E_Q}|\langle u, \frac{\xi}{\|\xi\|}\rangle|$$

is termed the zonoid of μ, see [30, Sec. 3.5]. It is well known that an even finite measure on the unit sphere is uniquely determined by its zonoid, see [30, Th. 3.5.3]. Therefore, if two measures share the same zonoid then the integrals of any even and integrable function with respect to them coincide.

For $f \in \mathcal{H}_e$ we have $f(0) = 0$ and therefore

$$\mathbf{E}f(\xi) = \mathbf{E}[f(\xi)\,\mathbb{1}_{\|\xi\|\neq 0}] = \mathbf{E_Q}f(\xi/\|\xi\|) = \int_{\mathbb{S}^{d-1}} f(u)\mu(du)\,.$$

Hence $\mathbf{E}f(\xi) = \mathbf{E}f(\xi^*)$ for each $f \in \mathcal{H}_e$. A short calculation shows that integrability of $f(\xi/\|\xi\|))$ under $\mathbf{Q}$ implies integrability of $f(\xi^*/\|\xi^*\|)$ under $\mathbf{Q}^*$ and vice versa. □

If ξ and ξ^* are zonoid equivalent, then $f(\xi)$ and $f(\xi^*)$ are two zonoid equivalent random variables for all $f \in \mathcal{H}_e$. The following result is easily derived by observing that $\mathbf{E}f(\xi) = \mathbf{E}\frac{1}{2}(f(\xi) + f(-\xi))$ for symmetric ξ.

Corollary 2.3. *Two symmetric random vectors ξ and ξ^* are zonoid equivalent if and only if $\mathbf{E}f(\xi) = \mathbf{E}f(\xi^*)$ for all $f \in \mathcal{H}$. In particular, $\mathbf{E}h_K(\xi) = \mathbf{E}h_K(\xi^*)$ for each convex compact set K.*

Corollary 2.4. *Let $f_1, \dots, f_k \in \mathcal{H}_e$. If ξ and ξ^* are zonoid equivalent, then the vectors $(f_1(\xi), \dots, f_k(\xi))$ and $(f_1(\xi^*), \dots, f_k(\xi^*))$ are zonoid equivalent as long as one of these vectors is integrable.*

Proof. It suffices to use the fact that $f(x) = |u_1 f_1(x) + \dots + u_k f_k(x)|$ belongs to $\mathcal{H}_e$ and $f(\xi)$ is integrable. □

The following easy fact is also worth noticing.

Proposition 2.5. *Two random vectors are zonoid equivalent if and only if all their linear transformations are zonoid equivalent.*

Proof. For each matrix A, we have $\mathbf{E}|\langle A\xi, u\rangle| = \mathbf{E}|\langle \xi, A^\top u\rangle|$ and so $A\xi$ and $A\xi^*$ are zonoid equivalent if ξ and ξ^* are. □

In view of Proposition 2.5, it is possible to define zonoid equivalence for random elements in a Banach space by zonoid equivalence of all linear maps from the Banach space to $\mathbb{R}^d$.

In the following we often consider random vectors with positive coordinates (shortly called positive vectors), which are usually denoted by the letter η.

Proposition 2.6. *Two positive integrable random vectors η and η^* are zonoid equivalent if and only if $\mathbf{E}f(\eta) = \mathbf{E}f(\eta^*)$ for each $f \in \mathcal{H}$. In particular, the zonoid equivalence implies $\mathbf{E}\eta = \mathbf{E}\eta^*$.*

Proof. While the sufficiency is evident, the necessity can be proved similarly to Theorem 2.2 with $\mathbf{Q}$ having density $\eta_1/\mathbf{E}\eta_1$. The equality of expectations is obtained by setting $f(x) = (x_i)_+$ for any $i = 1, \dots, d$. □

For positive random vectors, the concept of a *max-zonoid* is also useful. The max-zonoid M_η of a positive random vector $\eta = (\eta_1, \dots, \eta_d)$ is defined as the expectation of the crosspolytope in $\mathbb{R}^d$, which is the convex hull of the origin and the standard basis vectors scaled by $\eta_1, \dots, \eta_d$, see [19]. The support function of M_η is given by

$$h_{M_\eta}(u) = \mathbf{E}\max(0, u_1\eta_1, \dots, u_d\eta_d)\,, \quad u = (u_1, \dots, u_d) \in \mathbb{R}^d\,. \tag{2.1}$$

This support function is most interesting for positive $u_1, \dots, u_d$, where it is possible to omit zero in the right-hand side of (2.1).

Proposition 2.7. *Two positive integrable random vectors η and η^* have identical max-zonoids if and only if η and η^* are zonoid equivalent.*

Proof. It is shown in [23] that the equality of max-zonoids of positive random vectors is equivalent to the equality of their zonoids, which implies the zonoid equivalence. If η and η^* are zonoid equivalent, then their expectations coincide by Proposition 2.6 and so $Z_\eta = Z_{\eta^*}$, whence the max-zonoid are also identical by [23]. $\square$

3 Zonoid equivalence and isometries

An integrable random vector ξ in $\mathbb{R}^d$, which is not a.s. zero, generates a norm on $\mathbb{R}^d$ by

$$\|u\|_\xi = \mathbf{E}|\langle u, \xi\rangle| \,.$$

With this definition, zonoid equivalence of ξ and ξ^* means that $(\mathbb{R}^d, \|\cdot\|_\xi)$ and $(\mathbb{R}^d, \|\cdot\|_{\xi^*})$ are isometric.

A result of Hardin [12, Th. 1.1] reformulated for random vectors implies that, for any given positive $p \notin 2\mathbb{Z}$, the values $\mathbf{E}|1 + \langle u, \xi\rangle|^p$ for all $u \in \mathbb{R}^d$ determine uniquely the distribution of random vector $\xi \in \mathbb{R}^d$. If $p = 1$, this result means that the centred lift zonoid of ξ uniquely identifies the distribution of ξ, cf. [17, 24]. This also means that if two zonoid equivalent random vectors contain the same coordinate being exactly one, then the isometry of the corresponding finite-dimensional spaces implies that random vectors are identically distributed. Below we provide a generalisation of this result for $p = 1$ and symmetric random vectors showing that it is possible to replace the constant with a random variable.

Theorem 3.1. *Let ξ be a symmetric random vector in $\mathbb{R}^d$. If ε is any random variable with values ± 1, then the centred zonoid of (ε, ξ), i.e. the values of*

$$\mathbf{E}|u_0\varepsilon + \langle u, \xi\rangle| \,, \quad u_0 \in \mathbb{R},\ u \in \mathbb{R}^d \,,$$

determines uniquely the distribution of ξ.

Proof. For each function $f(\varepsilon, \xi)$ we have $f(\varepsilon, \xi) + f(-\varepsilon, \xi) = f(1, \xi) + f(-1, \xi)$, so that

$$\mathbf{E}|u_0\varepsilon + \langle u, \xi\rangle| + \mathbf{E}| - u_0\varepsilon + \langle u, \xi\rangle| = \mathbf{E}|u_0 + \langle u, \xi\rangle| + \mathbf{E}| - u_0 + \langle u, \xi\rangle| \,.$$

Since ξ is symmetric,

$$\mathbf{E}| - u_0 + \langle u, \xi\rangle| = \mathbf{E}|u_0 + \langle u, -\xi\rangle| = \mathbf{E}|u_0 + \langle u, \xi\rangle| \,.$$

Thus,

$$\mathbf{E}|u_0 + \langle u, \xi\rangle| = \frac{1}{2}(\mathbf{E}|u_0\varepsilon + \langle u, \xi\rangle| + \mathbf{E}| - u_0\varepsilon + \langle u, \xi\rangle|)$$

for all $u_0 \neq 0$ and $u \in \mathbb{R}^n$. Therefore, the right-hand side is determined by the zonoid of (ε, ξ), and it remains to note that the left-hand side uniquely identifies the distribution of ξ by [12, Th. 1.1]. $\square$

The uniqueness result in [12] is used to characterise isometries of subspaces of L^p that contain function identically equal one. Theorem 3.1 makes it possible to obtain similar results for subspaces of L^1 that consist of symmetric random variables and contain a random variable taking values ± 1. It is clearly possible to replace ± 1 with $\pm c$ for any fixed $c > 0$.

The isometries and zonoid equivalence can also be studied for collections of random elements, most notably for stochastic processes.

Definition 3.2. Two families of integrable random variables $\{\xi_t,\ t \in T\}$ and $\{\xi_t^*,\ t \in T\}$ are called *zonoid equivalent* if all their finite-dimensional distributions are zonoid equivalent, i.e.

$$\mathbf{E}|u_1\xi_{t_1} + \cdots + u_n\xi_{t_n}| = \mathbf{E}|u_1\xi_{t_1}^* + \cdots + u_n\xi_{t_n}^*| \tag{3.1}$$

for all $n \geq 1$, $t_1, \ldots, t_n \in T$ and $u_1, \ldots, u_n \in \mathbb{R}$.

A collection of integrable random elements $\{\xi_t,\ t \in T\}$ is a subset of the space $L^1 = L^1(\Omega, \mathfrak{K}, \mathbf{P})$. Denote by F_ξ the L^1-closure of the linear space generated by this collection. Assume that Ω is a Borel space with $\mathfrak{K}$ being the Borel σ-algebra.

Assume that $\{\xi_t\}$ is rigid, i.e. any linear isometry $U_0 \colon F_\xi \mapsto L^1$ is uniquely extended to the isometry $U \colon L^1 \mapsto L^1$. It is well known [12, 26] that the rigidity is guaranteed by imposing that the random elements $\{\xi_t\}$ have full support, i.e. the union of its supports is Ω up to a null set, and that $\xi_t/\bar{\xi}$, $t \in T$, generate the σ-algebra $\mathfrak{K}$, where $\bar{\xi} \in F_\xi$ is a random variable with full support.

Consider another rigid collection $\{\xi_t^*,\ t \in T\}$, which is zonoid equivalent to $\{\xi_t,\ t \in T\}$. Then the isometry between F_ξ and F_{ξ^*} can be characterised as follows, see Theorem 3.2 in [26]. For every $t \in T$,

$$\xi_t^*(\omega) = h(\omega)\xi_t(\phi(\omega)) \quad \mathbf{P}\text{-a.s.}, \tag{3.2}$$

where $\phi : \Omega \to \Omega$, $h : \Omega \to \mathbb{R}\backslash\{0\}$ are measurable and $|\bar{\xi}| d\mathbf{P} = |\bar{\xi}|(|h| d\mathbf{P}) \circ \phi^{-1}$, for a random variable with full support $\bar{\xi} \in F_\xi$.

If both F_ξ and F_{ξ^*} are symmetric and contain a random variable with values ± 1, then the existence of an isometry means that the finite-dimensional distributions of ξ and ξ^* coincide. In general, one has the following result for possibly non-rigid collections.

Corollary 3.3 (Cor. 4.2, [26])**.** *Let F_ξ be rigid and let F_{ξ^*} be zonoid equivalent to F_ξ and let F_{ξ^*} has the full support. Then for every $t \in T$,*

$$\xi_t^*(\omega) = h(\omega)\xi_t(\phi(\omega)) \quad \mathbf{P}\text{-a.s.},$$

where $\phi : \Omega \to \Omega$ and $h : \Omega \to \mathbb{R}\backslash\{0\}$ are measurable and $\mathbf{P}$ is equivalent to $\mathbf{P} \circ \phi^{-1}$.

A similar constriction of isometries can be carried over for max-zonoids and non-negative integrable functions, see [9], where such isometries are called pistons. Since for positive random vectors the zonoid equivalence and the max-zonoid equivalence are identical (see Proposition 2.7), the isometries corresponding to max-zonoids are also characterised by (3.2). The isometries are especially useful if the zonoid (max-zonoid) of a random element is invariant with respect to some transformation as described in the following sections.

4 Zonoid stationarity and stationary stable processes

The properties of linear isometries defined on families of random variables are important for the studies of symmetric stable laws, see [12, 13, 26]. Recall that a *symmetric α-stable* random vector X in $\mathbb{R}^d$ with $\alpha \in (0,2)$ can be represented as the LePage series

$$X = \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \xi^k , \tag{4.1}$$

where $\Gamma_k = \zeta_1 + \cdots + \zeta_k$ are successive sums of i.i.d. standard exponential random variables and $\xi, \xi^1, \xi^2, \dots$ are i.i.d. integrable symmetric random vectors. Note that the ξ's are often assumed to be distributed on the unit sphere with an extra normalisation constant in front of the sum, see [27, Cor. 1.4.3]. The LePage series can be considered in much more general spaces, e.g. its variant for random vectors in $\mathbb{R}^d_+$ and with the coordinate-wise maximum instead of the sum yields max-stable random vectors, see [11] and [7] for even more general settings. If ξ is distributed on the unit sphere, its distribution is uniquely determined by the distribution of X. Otherwise the following result characterises possible ξ.

Theorem 4.1. *Two LePage series X and X^* given by (4.1) with $\alpha = 1$ and integrable symmetric summands distributed as ξ and ξ^* coincide in distribution if and only if ξ and ξ^* are zonoid equivalent.*

Proof. The points $\{(\Gamma_k^{-1}, \xi^k),\ k \geq 1\}$ build the Poisson point process on $(0,\infty)$ with intensity t^{-2}, $t > 0$, and independent marks ξ^k, $k \geq 1$. The formula for the probability generating functional of the marked Poisson process (see [6]) yields the characteristic function of X

$$\begin{aligned}
\mathbf{E} e^{\imath \langle u, X\rangle} &= \exp\{-\int_0^\infty \mathbf{E}(1 - e^{\imath t \langle u,\xi\rangle}) t^{-2} dt\} \\
&= \exp\{-\int_0^\infty \mathbf{E}(1 - \cos(t\langle u,\xi\rangle)) t^{-2} dt\} = \exp\{-\frac{\pi}{2} \mathbf{E}|\langle u,\xi\rangle|\} ,
\end{aligned}$$

since $\int_0^\infty (1-\cos(s)) s^{-2} ds = \pi/2$, where $\imath$ denotes the imaginary unit. Thus, the distribution of X is determined by $\mathbf{E}|\langle u,\xi\rangle|$, $u \in \mathbb{R}^d$. $\square$

Example 4.2. Since the same factors Γ_k enter different coordinates of X given by (4.1), one needs highly dependent coordinates of ξ in order to ensure that X has independent coordinates. For example, on the probability space $\Omega = [0,1]$ with the Lebesgue measure define $\xi_i = d\, \mathbb{1}_{\omega \in [(i-1)/d, i/d)}$, $i = 1, \dots, d$. Then

$$\mathbf{E}|u_1\xi_1 + \cdots + u_d \xi_d| = \sum_{i=1}^{d} |u_i| ,$$

so X given by (4.1) is composed of i.i.d. (scaled) Cauchy random variables.

The zoonoid equivalence of random vectors with positive components is closely related to the representation of max-stable laws. Let $\eta, \eta^1, \eta^2, \dots$ be i.i.d. copies of integrable random vector η with positive coordinates. Each max-stable random vector Y with the unit Fréchet distributed marginals can be represented as

$$Y = \max_{k\geq 1} \Gamma_k^{-1} \eta^k \tag{4.2}$$

for the sequence Γ_k defined above, see [11].

Theorem 4.3. *Two random vectors Y and Y^* generated by* (4.2) *with η and η^* coincide in distribution if and only if η and η^* are zonoid equivalent.*

Proof. It follows from [8] that

$$\mathbf{P}(Y_1 \leq y_1, \dots, Y_d \leq y_d) = \exp\left\{- \mathbf{E} \max_{i=1,\dots,n} \frac{\eta_i}{y_i}\right\} \tag{4.3}$$

for all $y_1, \dots, y_d > 0$. The expression in the exponential defines the support function of the max-zonoid of $(\eta_1, \dots, \eta_d)$. It remains to recall that the equality of max-zonoids is equivalent to the zonoid equivalence for positive random vectors, see Proposition 2.7. □

LePage series (4.1) and (4.2) with ξ and η being stochastic processes define stable and max-stable stochastic processes. For instance, the process Y given by

$$Y_t = \max_{i\geq 1} \Gamma_i^{-1} \eta_t^i \,, \qquad t \in T \,, \tag{4.4}$$

is called the *Brown–Resnick* process associated to η, where η is an integrable positive process and $\{\eta^k\}$ its i.i.d. copies. It is well known that Y is a simple max-stable process, i.e. for all $n \geq 1$ the pointwise maximum of $Y^1, \dots, Y^n$ being i.i.d. copies of Y has the same distribution as nY. A remarkable result of [8] says that for every stochastically continuous simple max-stable process Y can be represented as (4.4).

Corollary 4.4. *Two Brown–Resnick processes Y and Y^* associated to positive stochastic processes η and η^* respectively are identically distributed if and only if η and η^* are zonoid equivalent.*

Let $\{\xi_t,\ t \in T\}$ be a stochastic process such that ξ_t is absolutely integrable for all $t \in T$, where T is either integer grid $\mathbb{Z}^d$ or $\mathbb{R}^d$.

Definition 4.5. The process $\{\xi_t,\ t \in T\}$ is called *zonoid stationary* if $\{\xi_t,\ t \in T\}$ and $\{\xi_{t+s},\ t \in T\}$ are zonoid equivalent for all $s \in T$.

Obviously all integrable stationary processes are zonoid stationary. If both ξ and ξ^* are centred Gaussian processes, then by Corollary 7.7 their zonoid equivalence implies the equality of all finite-dimensional distributions, so then the zonoid stationarity is equivalent to the conventional stationarity. The same holds for symmetric α-stable processes with given $\alpha > 1$.

The fact that zonoid does not uniquely determine the general underlying distribution suggests that there exist non-stationary but zonoid stationary processes. The uniqueness is often lost by taking exponentials, as it is the case with log-infinitely divisible distributions in Section 7.2. The following result immediately follows from Theorem 7.8.

Proposition 4.6. *The process e^{ξ} is zonoid stationary if and only if, for all $s, t_1, \dots, t_n \in T$ and $n \geq 1$, the characteristic function of the finite-dimensional distributions of ξ satisfies*

$$\varphi_{(\xi_{t_1},\dots,\xi_{t_n})}(u - \imath w) = \varphi_{(\xi_{t_1+s},\dots,\xi_{t_n+s})}(u - \imath w)\,, \tag{4.5}$$

for all $u \in \mathbb{R}^n$ such that $\sum u_i = 0$ and for at least one (and then necessarily for all) w, such that $\sum w_k = 1$ and both sides in (4.5) are finite.

Proposition 4.7 (cf. [14])**.** *Process e^{ξ_t}, where ξ_t is Gaussian with mean μ_t and variance σ_t^2, is zonoid stationary if and only if $\xi_t - \mu_t$ has stationary increments and $\mu_t + \frac{1}{2}\sigma_t^2$ is constant for all t.*

Proof. Apply Corollary 7.10 noticing that the expectation of e^{ξ_t} remains constant and the increments are Gaussian with covariance matrix given by the variogram. □

Example 4.8. The Geometric Brownian motion $e^{W_t - t/2}$, where W_t is a Brownian motion, is zonoid stationary. More generally, if $\{\xi_t,\ t \geq 0\}$ is a Lévy process such that $\mathbf{E}e^{\xi_t} = e^{t\psi(1)} < \infty$ then $e^{\xi_t - t\psi(1)}$ is zonoid stationary, cf. [31].

The following result follows from Theorems 4.1 and 4.3.

Theorem 4.9. *A symmetric 1-stable process (respectively max-stable process with unit Fréchet marginals) is stationary if and only if it all its LePage (respectively Brown–Resnick) representations involve zonoid stationary process ξ (respectively η).*

If the max-stable process Y given by (4.4) is stationary, the process $\xi = \log \eta$ is called *Brown–Resnick stationary*, see [14]. Theorem 4.9 shows that the Brown–Resnick stationarity of ξ is equivalent to the zonoid stationarity of e^{ξ}.

For a zonoid stationary process ξ the spaces generated by $\{\xi_t, t \in T\}$ and $\{\xi_{t+h}, t \in T\}$ are isometric for all $h \in T$. This gives rise to a representation of ξ in term of isometries. Following [25], a measurable function $\phi : \Omega \times T \to \Omega$ is said to be a measurable flow if $\phi_{t_1+t_2}(\omega) = \phi_{t_1}(\phi_{t_2}(\omega))$ and $\phi_0(\omega) = \omega$ for all $t_1, t_2 \in T$ and $\omega \in \Omega$. The flow ϕ is said to be nonsingular if $P \circ \phi_t^{-1} \sim P$ for all $t \in T$. A measurable function $r : \Omega \times T \to \mathbb{R}$ is said to be a cocycle for a measurable flow ϕ if $r_{t_1+t_2}(\omega) = r_{t_1}(\omega) r_{t_2}(\phi_{t_1}(\omega))$ for all $t_1, t_2 \in T$ and for P-almost all $\omega \in \Omega$. The following result can be proved by replicating the proof of Theorem 3.1 from [25].

Theorem 4.10. *Let ξ be a zonoid stationary process and assume that F_ξ is rigid. Then*

$$\xi_t(\omega) = r_t(\omega) \left(\frac{dP \circ \phi_t}{dP}\right)(\omega)(\xi_0 \circ \phi_t)(\omega) \quad P\text{-a.s.}, \tag{4.6}$$

where $\{\phi_t, t \in T\}$ is a measurable nonsingular flow and $\{r_t, t \in T\}$ is a cocycle for ϕ taking values in $\{-1, 1\}$.

5 Swap-invariant sequences

A finite or infinite random sequence $\xi = (\xi_1, \xi_2, \dots)$ of random elements is said to be *exchangeable* if its distribution is invariant under finite permutations, i.e. the distribution of any finite subsequence is invariant under any permutation of its elements, see e.g. [16, Sec 1.1].

Definition 5.1. An integrable random vector is called *swap-invariant* if all random vectors obtained by permutations of its coordinates are zonoid equivalent. A sequence of integrable random variables is called swap-invariant if all its finite subsequences are swap-invariant.

An integrable random vector ξ with strictly positive components exhibiting the swap-invariance property restricted to permutation of its two components ξ_i and ξ_j is called ij-swap-invariant. This weaker variant of the swap-invariance property has been already introduced and applied in a financial context in [22].

The swap-invariance property of ξ immediately implies that $\mathbf{E}|\xi_1| = \cdots = \mathbf{E}|\xi_d|$. It is obvious that the exchangeable sequence is swap-invariant. The following examples show that the swap-invariance is weaker than the exchangeability property.

Example 5.2 (see [5])**.** On the probability space $\Omega = [0, 1]$ with the Lebesgue measure define

$$\xi_n = n(n+1)\, \mathbb{1}_{\omega \in ((n+1)^{-1}, n^{-1}]}\,, \quad n \geq 1\,. \tag{5.1}$$

By a direct computation it is easy to see that

$$\mathbf{E}|u_1\xi_1 + \cdots + u_n\xi_n| = \sum_{i=1}^{n} |u_i|\,,$$

so that the sequence is indeed swap-invariant, but not exchangeable. Further examples of this type can be obtained for general sequences of non-negative random variables with equal expectations and disjoint supports.

Corollary 7.10 implies the swap-invariance property of the following example. Since the sequence is not exchangeable the construction provides an example of swap-invariant non-exchangeable sequence of lognormal random variables.

Example 5.3. Let $Z_1, Z_2, \dots$ be a sequence of i.i.d. standard normal random variables and let $\{b_k,\ k \geq 1\}$ be a sequence of real numbers such that $\sum b_k^2 < \infty$. Define $\eta_i = e^{\xi_i}$, $i \geq 1$, where

$$\xi_i = Z_i + \sum_{k=1}^{\infty} b_k Z_k + \mu_i\,,$$

and

$$\mu_i = -\frac{1}{2}\mathrm{Var}(\xi_i) = -\frac{1}{2}(1 + \sum_{k=1}^{\infty} b_k^2 + 2b_i)\,.$$

By Corollary 7.10, η is swap-invariant. Note that no two components η_i and η_j are identically distributed unless $b_i = b_j$.

If the extended sequence $(1, \xi)$ (or (ε, ξ) with $\varepsilon \in 1\{-1, 1\}$ and symmetric ξ) is swap-invariant, then ξ is exchangeable. Actually, the swap-invariance of such extended sequence is stronger than the exchangeability of ξ, see Section 6.

It is well known that each exchangeable sequence of integrable random variables satisfies several ergodic theorems. Given an infinite random sequence $\{\xi_n,\ n \geq 1\}$, we denote the corresponding tail σ-algebra by $\mathcal{T}_\xi$, the shift-invariant σ-algebra by $\mathcal{I}_\xi$, and the permutation-invariant σ-field by $\mathcal{E}_\xi$. These σ-algebras are a.s. identical for exchangeable sequences, see [16, Cor. 1.6]. Since an infinite exchangeable sequence is stationary, the following result is a direct consequence of [15, Th. 10.6] and [16, Cor. 1.6].

Theorem 5.4. *Let $\xi_1, \xi_2, \dots$ be an exchangeable sequence of integrable random variables. Then*

$$n^{-1}\sum_{i=1}^{n} \xi_i \to \mathbf{E}(\xi_1 \mid \mathcal{E}_\xi) \quad \text{a.s. and in } L^1 \text{ as } n \to \infty. \tag{5.2}$$

In the following we extend this fact to swap-invariant sequences. Recall that these sequences by definition consist of integrable random variables.

Theorem 5.5. *Let $\xi_1, \xi_2, \dots$ be a swap-invariant sequence of random variables. Then $n^{-1}(\xi_1 + \dots + \xi_n)$ converges almost surely to an integrable random variable X as $n \to \infty$.*

Proof. Assume first that all random variables $\xi_1, \xi_2, \dots$ are symmetric and that at least one random variable (say ξ_1) is non-zero with probability one. Recall that $\mathbf{E}|\xi_i|$ is the same for all i. Define an equivalent to $\mathbf{P}$ probability measure $\mathbf{P}^1$ by

$$\frac{d\mathbf{P}^1}{d\mathbf{P}} = \frac{|\xi_1|}{\mathbf{E}|\xi_1|}\,. \tag{5.3}$$

For any finite subsequence $\xi = (\xi_1, \xi_{k_1} \dots, \xi_{k_d})$,

$$\frac{\mathbf{E}|\langle u, \xi\rangle|}{\mathbf{E}|\xi_1|} = \mathbf{E}_{\mathbf{P}^1}|u_1\varepsilon + u_2\frac{\xi_{k_1}}{|\xi_1|} + \dots + u_d\frac{\xi_{k_d}}{|\xi_1|}|\,, \tag{5.4}$$

where $\varepsilon = \xi_1/|\xi_1|$ is the sign of ξ_1 and $\mathbf{E}_{\mathbf{P}^1}$ denotes the expectation with respect to $\mathbf{P}^1$. By Theorem 3.1, the right-hand side of (5.4) determines the distribution of $(\xi_{k_1}, \dots, \xi_{k_d})/|\xi_1|$ under $\mathbf{P}^1$. By writing (5.4) for a permutation $\xi_{k_{i_1}}, \dots, \xi_{k_{i_d}}$ we arrive at the conclusion that the sequence $\frac{\xi_2}{|\xi_1|}, \frac{\xi_3}{|\xi_1|}, \dots$ is exchangeable under $\mathbf{P}^1$. Theorem 5.4 yields that

$$\frac{1}{n}\left(\frac{\xi_2}{|\xi_1|} + \cdots + \frac{\xi_n}{|\xi_1|}\right) \to Z \qquad \mathbf{P}^1\text{-a.s. as } n \to \infty$$

for some random variable Z. Since $\mathbf{P}^1$ and $\mathbf{P}$ are equivalent, the same holds $\mathbf{P}$-a.s. Thus,

$$\frac{\xi_2 + \cdots + \xi_n}{n} \to X = |\xi_1| Z \qquad \text{a.s. as } n \to \infty\,.$$

It is obviously possible to add ξ_1 in the numerator without altering the limit.

If the sequence $\{\xi_n\}$ is no longer symmetric, consider an independent symmetric random variable ε with values ± 1. Then the sequence $\{\varepsilon\xi_n,\ n \geq 1\}$ is symmetric and swap-invariant, which is seen by the total probability formula. As shown above, $\{\varepsilon\xi_n\}$ satisfies the ergodic theorem with limit X. Then the original sequence $\{\xi_n\}$ satisfies the ergodic theorem with the limit εX (note that ε and X may be dependent).

It remains to consider the case when all ξ_i have an atom at zero. Fix any $k \geq 1$ and define a new measure $\mathbf{P}^k$ by

$$\frac{d\mathbf{P}^k}{d\mathbf{P}} = \frac{|\xi_k|}{\mathbf{E}|\xi_k|}\,.$$

The function $(x_1, \dots, x_d) \mapsto |u_1 x_1 + \cdots + u_d x_d|\, \mathbb{1}_{x_k \neq 0}$ is in $\mathcal{H}_e$, hence

$$\begin{aligned}\mathbf{E}|u_1\xi_1 + u_2\xi_2 + \cdots + u_k\xi_k + \cdots + u_d\xi_d|\, \mathbb{1}_{\xi_k \neq 0} \\ = \mathbf{E}|u_1\xi_1 + u_2\xi_{i_2} + \cdots + u_k\xi_k + \cdots + u_n\xi_{i_d}|\, \mathbb{1}_{\xi_k \neq 0}\end{aligned}$$

for all $u_1, \dots, u_d \in \mathbb{R}$ and all permutations $i_1, \dots, i_d$ with $i_k = k$. Thus, the sequence $(\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots)/|\xi_k|$ is exchangeable under $\mathbf{P}^k$. Since $\mathbf{P}^k$ is equivalent to $\mathbf{P}$ restricted on $\{\xi_k \neq 0\}$, $n^{-1}(\xi_1 + \cdots + \xi_n)$ converges to some random variable X for almost all $\omega \in \{\xi_k \neq 0\}$. Note that the same limit appears under $\mathbf{P}^m$ for $m \neq k$ for almost all ω such that $\xi_k(\omega) \neq 0$ and $\xi_m(\omega) \neq 0$. Finally set $X(\omega) = 0$ for all $\omega \in \Omega$ such that $\xi_n(\omega) = 0$ for all $n \geq 1$.

Since $\xi_1, \dots, \xi_n$ are zonoid equivalent, the triangle inequality yields

$$\mathbf{E}\frac{1}{n}|\xi_1 + \cdots + \xi_n| \leq \frac{1}{n}(\mathbf{E}|\xi_1| + \cdots + \mathbf{E}|\xi_n|) = \mathbf{E}|\xi_1|\,.$$

By Fatou's Lemma

$$\mathbf{E}|X| \leq \liminf_{n\to\infty} \mathbf{E}\frac{1}{n}|\xi_1 + \cdots + \xi_n| \leq \mathbf{E}|\xi_1|\,,$$

which confirms the integrability of X. □

Remark 5.6. A proof of Theorem 5.5 for almost surely positive swap-invariant sequences can be alternatively carried over by using ξ_1 to change the measure and then referring to the unique characterisation of a random vector by its lift zonoid.

Theorem 5.7. *Assume that a swap-invariant sequence $\xi_1, \xi_2, \dots$ satisfies one of the following conditions:*

(a) $\xi_k \neq 0$ *a.s. for some* $k \geq 1$,

(b) ξ *is uniformly integrable.*

Then the convergence of $n^{-1}(\xi_1 + \dots + \xi_n) \to X$ *also holds in* L^1.

Proof. (a) The proof of Theorem 5.5 and Theorem 5.4 yield that

$$\mathbf{E}|n^{-1}(\xi_1 + \dots + \xi_n) - X|\, \mathbb{I}_{\xi_k \neq 0} \to 0 \qquad \text{as } n \to \infty\,,$$

while $\mathbf{P}(\xi_k \neq 0) = 1$.
(b) An equivalent condition to uniform integrability is that $\sup_{i\geq 1} \mathbf{E}|\xi_i| < \infty$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all measurable A with $\mathbf{P}(A) \leq \delta$ we have $\sup_{i\geq 1} \mathbf{E}|\xi_i|\, \mathbb{I}_A < \varepsilon$, see [15, Lemma 4.10]. Define $X_n = n^{-1}(\xi_1 + \dots + \xi_n)$. For ε, δ, A as above

$$\sup_{n\geq 1} \mathbf{E}|X_n|\, \mathbb{I}_A \leq \sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} \mathbf{E}|\xi_i|\, \mathbb{I}_A \leq \varepsilon\,,$$

which, together with $\mathbf{E}|X_n| \leq \mathbf{E}|\xi_1|$, show that the sequence $\{X_n\}$ is uniform integrable too. The a.s. convergence implies convergence in probability and so the L^1-convergence in view of the uniform integrability property, see [15, Prop. 4.12]. □

Example 5.8 (Example 5.2 continuation)**.** For the sequence (5.1), $n^{-1}(\xi_1 + \dots + \xi_n) \to 0$ a.s., but $\mathbf{E}n^{-1}(\xi_1 + \dots + \xi_n) = 1$, so the ergodic theorem holds almost surely but not in L^1.

The following theorem characterises the limits in Theorem 5.5 for the case when at least one random variable in the sequence does not have an atom at zero.

Theorem 5.9. *Let* $\xi = (\xi_1, \xi_2, \dots)$ *be a symmetric swap-invariant sequence such that* $\xi_1 \neq 0$ *a.s. Then*

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i \to \frac{|\xi_1|}{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})} \mathbf{E}(\xi_2 \mid \mathcal{E}_{\tilde{\xi}}) \quad \text{a.s. and in } L^1 \text{ as } n \to \infty, \tag{5.5}$$

where $\tilde{\xi} = (\xi_2/|\xi_1|, \xi_3/|\xi_1|, \dots)$.

Proof. The sequence $\tilde{\xi}$ is exchangeable under $\mathbf{P}^1$ defined by (5.3) and Theorem 5.4 implies

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\xi_i}{|\xi_1|} \to \mathbf{E}_{\mathbf{P}^1} \left[\frac{\xi_2}{|\xi_1|} \mid \mathcal{E}_{\tilde{\xi}} \right] \quad \text{a.s. and in } L^1 \text{ as } n \to \infty. \tag{5.6}$$

Let Z be a $\mathcal{E}_{\tilde{\xi}}$ measurable and $\mathbf{P}^1$-integrable random variable. Then

$$\mathbf{E}_{\mathbf{P}^1} Z = \mathbf{E}\frac{|\xi_1| Z}{\mathbf{E}|\xi_1|} = \mathbf{E}\left[\mathbf{E}\big(\frac{|\xi_1| Z}{\mathbf{E}|\xi_1|} \mid \mathcal{E}_{\tilde{\xi}}\big)\right] = \mathbf{E}\left[Z\frac{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})}{\mathbf{E}|\xi_1|}\right] . \tag{5.7}$$

Let $A \in \mathcal{E}_{\tilde{\xi}}$. By the definition of the conditional expectation

$$\begin{aligned}\mathbf{E}_{\mathbf{P}^1}\big(\mathbb{I}_A \, \mathbf{E}_{\mathbf{P}^1}\big(\frac{\xi_2}{|\xi_1|} \mid \mathcal{E}_{\tilde{\xi}}\big)\big) &= \mathbf{E}(\mathbb{I}_A \, \xi_2/\mathbf{E}|\xi_1|) = \mathbf{E}(\mathbb{I}_A \, \mathbf{E}(\xi_2/\mathbf{E}|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})) \\ &= \mathbf{E}\left[\mathbb{I}_A \, \frac{\mathbf{E}(\xi_2 \mid \mathcal{E}_{\tilde{\xi}})}{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})} \frac{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})}{\mathbf{E}|\xi_1|}\right] = \mathbf{E}_{\mathbf{P}^1}\left[\mathbb{I}_A \, \frac{\mathbf{E}(\xi_2 \mid \mathcal{E}_{\tilde{\xi}})}{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})}\right] ,\end{aligned}$$

where the last equality follows from (5.7). The uniqueness of the conditional expectation yields

$$\mathbf{E}_{\mathbf{P}^1}\left[\frac{\xi_2}{|\xi_1|} \mid \mathcal{E}_{\tilde{\xi}}\right] = \frac{\mathbf{E}(\xi_2 \mid \mathcal{E}_{\tilde{\xi}})}{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\tilde{\xi}})} \quad \text{a.s.}$$

This equation together with (5.6) yield the claim. □

With a similar proof we arrive at the following result for positive sequences.

Proposition 5.10. *Let $\xi = (\xi_1, \xi_2, \dots)$ be a positive swap-invariant sequence. Then*

$$\frac{1}{n}\sum_{i=1}^{n} \xi_i \to \frac{\xi_1}{\mathbf{E}(\xi_1 \mid \mathcal{E}_{\tilde{\xi}})}\mathbf{E}(\xi_2 \mid \mathcal{E}_{\tilde{\xi}}) \quad \text{a.s. and in } L^1 \text{ as } n \to \infty, \tag{5.8}$$

where $\tilde{\xi} = (\xi_2/\xi_1, \xi_3/\xi_1, \dots)$.

For non-symmetric swap-invariant sequences we get the following result by applying the total probability formula and Theorem 5.9.

Corollary 5.11. *Let $\xi = (\xi_1, \xi_2, \dots)$ be a swap-invariant sequence such that $\xi_1 \neq 0$ a.s. Then*

$$\frac{1}{n}\sum_{i=1}^{n} \xi_i \to \frac{|\xi_1|}{\mathbf{E}(|\xi_1| \mid \mathcal{E}_{\varepsilon\tilde{\xi}})\varepsilon}\mathbf{E}(\varepsilon\xi_2 \mid \mathcal{E}_{\varepsilon\tilde{\xi}}) \quad \text{a.s. and in } L^1 \text{ as } n \to \infty, \tag{5.9}$$

where ε is the Rademacher random variable independent of ξ under $\mathbf{P}$.

Example 5.12 (Example 5.3 continuation)**.** We show that $n^{-1}(\eta_1 + \cdots + \eta_n)$ converges a.s. to

$$X = \exp\Big(\sum_{i=1}^{\infty} b_i Z_i - \frac{1}{2}\sum_{i=1}^{\infty} b_i^2\Big) .$$

We give two proofs, one direct and another using Proposition 5.10.

(a). Denote $\eta_i^* = \eta_i / X = \exp(Z_i - 1/2(1+2b_i))$. Since $\sum_{i=1}^\infty b_i^2 < \infty$, the sequence $\{b_i, i \geq 1\}$ is bounded and therefore

$$m_k = \min_{i \geq k} e^{-b_i} \quad \text{and} \quad M_k = \max_{i \geq k} e^{-b_i}$$

are well defined. Since $m_k e^{Z_i - 1/2} \leq \eta_i^* \leq M_k e^{Z_i - 1/2}$ for all $i \geq k$ and the i.i.d. bounding sequences obey the strong law of large numbers with limits m_k and M_k,

$$m_k \leq \lim_{n \to \infty} \frac{1}{n} (\eta_k^* + \cdots + \eta_{k+n}^*) \leq M_k \,.$$

Obviously, one can start the summation with η_1^* without altering the limit. Since $i \to \infty$, $b_i \to 0$ and so both m_k and M_k converge to one. By letting $k \to \infty$

$$\frac{1}{n}(\eta_1 + \cdots + \eta_n) = X \frac{1}{n}(\eta_1^* + \cdots + \eta_n^*) \to X \quad \text{as } n \to \infty \,.$$

(b). By [16, Cor. 1.6] we can consider the tail σ-field $\mathcal{T}_{\tilde{\eta}}$, where

$$\tilde{\eta} = (\frac{\eta_2}{\eta_1}, \frac{\eta_3}{\eta_1}, \dots) = (e^{Z_2 - Z_1 - (b_2 - b_1)}, e^{Z_3 - Z_1 - (b_3 - b_1)}, \dots) \,.$$

Since the functions $x \mapsto e^{x - (b_i - b_1)}$, $i \geq 2$, are bijective, $\mathcal{T}_{\tilde{\eta}}$ can be written as $\mathcal{T}_{\tilde{\eta}} = \bigcap_{n \geq 2} \mathcal{F}_n$, where $\mathcal{F}_n = \sigma(Z_n - Z_1, Z_{n+1} - Z_1, \dots)$. For each $n \geq 2$, the random variable

$$\tilde{Z}_n = \lim_{k \to \infty} k^{-1} \sum_{i=0}^{k-1} (Z_1 - Z_{n+i})$$

is clearly $\mathcal{F}_n$-measurable and by the strong law of large numbers $\tilde{Z}_n = Z_1$ a.s. Thus, Z_1 is measurable with respect to the completion $\bar{\mathcal{F}}_n$ of $\mathcal{F}_n$ for all $n \geq 2$, and hence $\bar{\mathcal{T}}_{\tilde{\eta}}$ measurable. On the other hand, for all $n \geq 2$, the vector $(Z_2, \dots, Z_n)$ is independent of $\mathcal{F}_{n+1}$ and therefore independent of $\mathcal{T}_{\tilde{\eta}}$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Then for all $A \in \mathcal{T}_{\tilde{\eta}}$, the dominated convergence theorem yields

$$\begin{aligned}
\mathbf{E}\, \mathbb{I}_A\, f(\sum_{i=2}^\infty b_i Z_i) &= \lim_{k \to \infty} \mathbf{E}\, \mathbb{I}_A\, f(\sum_{i=2}^k b_i Z_i) \\
&= \lim_{k \to \infty} \mathbf{P}(A) \mathbf{E} f(\sum_{i=2}^k b_i Z_i) = \mathbf{P}(A) \mathbf{E} f(\sum_{i=2}^\infty b_i Z_i) \,,
\end{aligned}$$

which shows the independence of $\sum_{i=2}^\infty b_i Z_i$ and $\mathcal{T}_{\tilde{\eta}}$. Since $\mathbf{E}(Z \mid \mathcal{T}_{\tilde{\eta}}) = \mathbf{E}(Z \mid \bar{\mathcal{T}}_{\tilde{\eta}})$ a.s. for all integrable Z,

$$\begin{aligned}
\mathbf{E}(\eta_1 \mid \mathcal{T}_{\tilde{\eta}}) &= e^{(1+b_1)Z_1} e^{-\frac{1}{2}(1 + b_1^2 + 2b_1)} \,, \\
\mathbf{E}(\eta_2 \mid \mathcal{T}_{\tilde{\eta}}) &= e^{b_1 Z_1} e^{-\frac{1}{2} b_1^2} \,.
\end{aligned}$$

By Proposition 5.10,

$$\frac{1}{n} \sum_{i=1}^n \eta_i \to \frac{X e^{Z_1} e^{-\frac{1}{2}(1+2b_1)}}{e^{(1+b_1)Z_1} e^{-\frac{1}{2}(1+b_1^2+2b_1)}} e^{b_1 Z_1} e^{-\frac{1}{2} b_1^2} = X \quad \text{a.s. and in } L^1 \text{ as } n \to \infty.$$

Corollary 5.13. *Let ξ be a swap-invariant sequence. If $n^{-1}(\xi_1+\cdots+\xi_n)$ converges in L^1 to a deterministic non-zero limit c, then (c,ξ) is swap-invariant and so ξ is exchangeable.*

Proof. For $m,n\geq 1$, the swap-invariance property implies

$$\mathbf{E}|u_1\xi_1+\cdots+u_n\xi_n+u_0\frac{1}{m}\sum_{k=1}^m \xi_{n+k}| = \mathbf{E}|u_{i_1}\xi_1+\cdots+u_{i_n}\xi_n+u_{i_0}\frac{1}{m}\sum_{k=1}^m \xi_{n+k}|\,,$$

for all permutations $(i_0,i_1,\dots,i_n)$ of $(0,1,\dots,n)$. The L^1-convergence then yields as $m\to\infty$

$$\mathbf{E}|u_0c+u_1\xi_1+\cdots+u_n\xi_n| = \mathbf{E}|u_{i_0}c+u_{i_1}\xi_1+\cdots+u_{i_n}\xi_n|\,,$$

so that (c,ξ) is swap-invariant. □

6 Non-centred zonoids

It is possible to relate the centred and non-centred zonoids as $Z^o_\xi=Z_\xi+Z_{-\xi}$, i.e. the centred zonoid is the Minkowski (elementwise) sum of the zonoid of ξ and the zonoid of $-\xi$. The latter is obtained as the central symmetry (with respect to the origin) of Z_ξ. If ξ has a symmetric distribution, then $Z^o_\xi=2Z_\xi$ is a scaled zonoid of ξ. For a general integrable ξ, its centred zonoid equals $2Z_{\varepsilon\xi}$, where ε is the Rademacher random variable taking values ±1 with equal probability and independent of ξ. Note that the conventional symmetrisation $\xi-\xi'$ for i.i.d. ξ and ξ' is not helpful in this context.

Proposition 6.1. *If ξ and ξ^* are two integrable random vectors, then $Z_\xi=Z_{\xi^*}$ if and only if $\mathbf{E}\xi=\mathbf{E}\xi^*$ and $Z^o_\xi=Z^o_{\xi^*}$.*

Proof. Since $2a_+=|a|+a$ for any real a,

$$h_{Z_\xi}(u)=\frac{1}{2}(\mathbf{E}|\langle\xi,u\rangle|+\langle\mathbf{E}\xi,u\rangle)\,.$$

It remains to note that the equality $Z_\xi=Z_{\xi^*}$ implies the equality of expectations by [24, Prop. 2.11]. □

In view of the above fact, Proposition 2.6 implies that for positive random vectors the equivalences of centred and non-centred zonoids are identical concepts.

Now consider lift zonoids and their centred variant. Since each of them determines uniquely the distribution of a random vector, the exchangeability of ξ is equivalent to the symmetry of the lift zonoid or its centred variant with respect to all hyperplanes

$$H_{ij}=\{(u_0,u_1,\dots,u_d)\in\mathbb{R}^{d+1}:\ u_i=u_j\}\,,\quad 1\leq i<j\leq d, \tag{6.1}$$

so that the symmetry operation corresponds to swapping the coordinates of ξ. The swap-invariance property of ξ is equivalent to the symmetry of the zonoid Z^o_ξ with respect to hyperplanes $\{u=(u_1,\dots,u_d)\in\mathbb{R}^d:\ u_i=u_j\}$, $1\leq i<j\leq d$.

There is also a natural geometric way to define a stronger property than exchangeability. Following [21] an integrable random vector is said to be *jointly self-dual* if its lift zonoid $\hat{Z}_\xi$ is symmetric with respect to all hyperplanes H_{ij} defined in (6.1) with $0 \le i < j \le d$ meaning that $\mathbf{E}(u_0 + u_1\xi_1 + \dots + u_d\xi_d)_+$ is invariant with respect to any permutation of $u_0, u_1, \dots, u_d$. This is stronger than the swap-invariance of $(1, \xi)$, meaning that $\mathbf{E}|u_0 + u_1\xi_1 + \dots + u_d\xi_d|$ is invariant with respect to any permutation of $u_0, u_1, \dots, u_d$. Still, for positive random vectors the joint self-duality and the swap-invariance of $(1, \xi)$ are the same.

While the joint self-duality implies the exchangeability, the converse is false, see [21]. For instance a vector of i.i.d. positive random variables is exchangeable, but is neither jointly self-dual nor is $(1, \xi)$ swap-invariant. A version of the self-duality property corresponding to the permutation of the lifting coordinate and *one* fixed other coordinate was studied in [21]. In particular its univariate version is often called *put-call symmetry* and is intensively discussed and applied in the financial literature, see e.g. [3, 32] and further literature cited in [21].

Proposition 6.2. *If a non-trivial random vector ξ is jointly self-dual, then all its components are almost surely positive random variables with expectation being one.*

Proof. It suffices to prove this for random variable ξ. The self-duality property of ξ implies that

$$\mathbf{E}(0 + (-1)\xi)_+ = \mathbf{E}(-1 + 0\xi)_+ \,,$$

so that $\mathbf{E}\xi_- = 0$ and so ξ is almost surely non-negative. Since

$$\mathbf{E}(0 + 1\xi)_+ = \mathbf{E}(1 + 0\xi)_+ \,,$$

it follows that $1 = \mathbf{E}\xi_+ = \mathbf{E}\xi$.

If ξ has an atom at zero, then $\mathbf{E}(1 - a\xi)_+$, $a \in \mathbb{R}$, is bounded from below by a positive number. The self-duality implies that $\mathbf{E}(-a + \xi)_+$ is also bounded from below by the same number, which is not possible for large a in view of the integrability of ξ. □

For integrable random vectors with positive components the symmetry properties can be related to each other. Following the notation of [22] we first define the family of functions,

$$\tilde{\kappa}_j(x) = \left(\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}\right) \,, \quad j = 1, \dots, d \,,$$

on $x \in (0, \infty)^d$. For any $j = 1, \dots, d$ define a new probability measure by

$$\frac{d\mathbf{P}^j}{d\mathbf{P}} = \frac{\eta_j}{\mathbf{E}\eta_j} \,. \qquad (6.2)$$

Consider an integrable random vector η with positive components. If $\mathbf{E}\eta_j = 1$, then the zonoid of η coincides with the lift zonoid of $\tilde{\kappa}_j(\eta)$ under $\mathbf{P}^j$, see [23, Prop. 3]. The above measure change was used in [10] in order to reduce the dimensionality when calculating option prices.

Theorem 6.3. *Assume that η is an integrable random vector of dimension $d \geq 2$ with positive components. The following conditions are equivalent*

(a) η is swap-invariant under $\mathbf{P}$.

(b) $\tilde{\kappa}_j(\eta)$ is jointly self-dual under $\mathbf{P}^j$.

(c) In case $d \geq 3$, for at least two $j \in \{1, \dots, d\}$ (and then automatically for all j), $\tilde{\kappa}_j(\eta)$ is exchangeable under $\mathbf{P}^j$.

Proof. The equivalence of *(a)* and *(b)* is obtained (for $j = 1$) by

$$\mathbf{E}|u_1\eta_1 + \dots + u_d\eta_d| = \mathbf{E}\eta_1 \mathbf{E}_{\mathbf{P}^1} \left| u_1 + u_2 \frac{\eta_2}{\eta_1} + \dots + u_d \frac{\eta_d}{\eta_1} \right|,$$

so that permutations of coordinates in the left hand side corresponds to permutations in the right hand side. The invariance with respect to the latter is equivalent to the exchangeability of $\tilde{\kappa}_1(\eta)$ under $\mathbf{P}^1$, since the right hand side identifies the distribution of $\tilde{\kappa}_1(\eta)$. Note that we have used the positivity of η.

It is easy to see that *(a)* implies *(c)* for all j, since the exchangeability is a weaker property than *(b)*. Assuming *(c)* for $j = 1, 2$ without loss of generality, we see that $(\eta_2/\eta_1, \dots, \eta_d/\eta_1)$ is $\mathbf{P}^1$-exchangeable and $(\eta_1/\eta_2, \eta_3/\eta_2, \dots, \eta_d/\eta_2)$ is $\mathbf{P}^2$-exchangeable. The first fact implies that $\mathbf{E}|\langle u, \eta \rangle|$ is invariant with respect to permutation all but first coordinates of u and the second fact implies the invariance with respect to permutations of all coordinates excluding the second one, so η is swap-invariant. □

7 Equality of zonoids

7.1 Location-scale families

Consider family of random variables $\xi = \mu + \sigma X$ for a integrable random variable X and $\mu \in \mathbb{R}$, $\sigma > 0$. These random variables are said to form a location-scale family.

Theorem 7.1. *Assume that the distribution of X has infinite essential infimum and essential supremum. Then the zonoid Z_ξ of a random variable ξ from the location-scale family generated by X uniquely determines the location and scale parameters of the distribution.*

Proof. Without loss of generality set $\mathbf{E}X = 0$. Assume that the random variables $\mu + \sigma X$ and $\mu^* + \sigma^* X$ share the same zonoid. By Proposition 6.1, $\mu = \mu^*$.

In order to finish the proof we show that $\mathbf{E}(\mu + \sigma X)_+$ is strictly increasing in σ for each fixed $\mu \in \mathbb{R}$. This is obvious if $\mu = 0$, since $\mathbf{E}(\sigma X)_+ = \sigma \mathbf{E}X_+$, which is strictly increasing in σ since $\mathbf{E}X_+ > 0$.

Assume that $\mu < 0$ and $\sigma_1 > \sigma_2$. Then

$$\begin{aligned} &\mathbf{E}((\mu + \sigma_1 X)_+ - (\mu + \sigma_2 X)_+) \\ &\qquad = \mathbf{E}((\mu + \sigma_1 X)\, \mathbb{1}_{\{-\frac{\mu}{\sigma_1} < X \leq -\frac{\mu}{\sigma_2}\}}) + (\sigma_1 - \sigma_2)\mathbf{E}(X\, \mathbb{1}_{\{-\frac{\mu}{\sigma_2} < X\}}) > 0, \end{aligned}$$

where the last expectation is strictly positive because X has unbounded support and $\mathbf{E}X = 0$.

If $\mu > 0$, then the same argument applied to $\mathbf{E}(\mu + \sigma_1 X)_-$ yields that the expectation of the negative part is strictly decreasing in σ and the equality $\mathbf{E}(\mu+\sigma_1 X)_+ = \mu - \mathbf{E}(\mu+\sigma_1 X)_-$ concludes the proof. □

Note that Theorem 7.1 does not hold for the centred zonoid Z_ξ^o unless it is assumed that the expectation of ξ is known and so Z_ξ is also identified.

Corollary 7.2. *Assume that random variable ξ has infinite essential infimum and essential supremum. If $Z_\xi = Z_{\sigma\xi+\mu}$, then $\mu = 0$ and $\sigma = 1$.*

Corollary 7.3. *Two normally distributed d-dimensional random vectors ξ and ξ^* coincide in distribution if and only if $Z_\xi = Z_{\xi^*}$.*

Proof. For $u \in \mathbb{R}^d$ the random variables $\langle \xi, u\rangle$ and $\langle \xi^*, u\rangle$ belong to the same location-scale family. The proof is finished by referring to Theorem 7.1 and noticing that all one-dimensional projection of a random vector uniquely determine its distribution. □

The uniqueness holds also for the location scale family obtained as $\mu+\sigma X$ for a symmetric stable random variable X.

Example 7.4 (Distribution with bounded support)**.** Assume that $\mathbf{E}X = 0$ and that X has finite essential infinum, i.e. there exists a constant c such that $X \geq c$ a.s. Choose $\mu > 0$. Then for all $\sigma < -\mu/c$ the random variable $\xi = \mu+\sigma X$ is a.s. positive and so the expectation of its negative part is zero and the expectation of its positive part is μ. Thus the zonoid Z_ξ does not uniquely determine the scale parameter σ.

Note that all above results are formulated for non-centred zonoids. In the rest of this section we consider centred zonoids, which are used to define the zonoid equivalence. The following result concerns random vectors that can be represented as product of a scaling random variable and an independent random vector.

Proposition 7.5. *Two random vectors $\xi = R\zeta$ and $\xi^* = R^*\zeta^*$, where R and R^* are positive random variables independent of ζ and ζ^* respectively, are zonoid equivalent if and only if $(\mathbf{E}R)\zeta$ and $(\mathbf{E}R^*)\zeta^*$ are zonoid equivalent.*

Proof. It suffices to note that

$$\mathbf{E}|\langle u, \xi\rangle| = \mathbf{E}R\mathbf{E}|\langle u, \zeta\rangle| = \mathbf{E}|\langle u, (\mathbf{E}R)\zeta\rangle| \,.$$

□

Consider random vectors with centred *elliptical* distributions, i.e. assume that $\xi = R(AU)$, where U is uniformly distributed on the unit sphere, A is a (deterministic) matrix and R is a positive random variable independent of U.

Proposition 7.6. *Two centred elliptically distributed random vectors $\xi = R(AU)$ and $\xi^* = R^*(A^*U)$ are zonoid equivalent if and only if* $(\mathbf{E}R)^2 AA^\top = (\mathbf{E}R^*)^2 A^*(A^*)^\top$.

Proof. Using rescaling, it is possible to assume that $\mathbf{E}R = \mathbf{E}R^*$. By Proposition 7.5, it suffices to consider zonoid equivalence of AU and A^*U. By Proposition 2.5, this is the case if and only if random variables $\langle A^\top u, U\rangle$ and $\langle (A^*)^\top u, U\rangle$ are zonoid equivalent. Since U is uniformly distributed on the unit sphere, $\langle v, U\rangle$ is distributed as a certain random variable with a fixed distribution scaled by $\|v\|$ for all v. Thus, $\|A^\top u\| = \|(A^*)^\top u\|$ for all u, which implies the statement. □

Corollary 7.7. *Two symmetric normally distributed random vectors ξ and ξ^* coincide in distribution if and only if they are zonoid equivalent.*

Zonoid of symmetric α-stable random vector ξ with $\alpha \in (1, 2]$ is computed in [20, Sec. 6.4] as

$$Z_\xi = \frac{1}{\pi}\Gamma(1 - \frac{1}{\alpha})K\,,$$

where Γ is the gamma-function and K is a convex body that, together with α, characterises the distribution of ξ. Thus, if α is given, then the zonoid determines uniquely the corresponding symmetric α-stable distribution. However, two symmetric stable vectors with the same zonoid are not necessarily identically distributed if their stability indices are different.

7.2 Log-infinitely divisible distributions with equal zonoids

A random vector with positive components can be written as the coordinate-wise exponential $\eta = e^\xi$. In the following φ_ξ stands for the characteristic function of ξ.

Theorem 7.8 (See [22] and [23])**.** *Two integrable random vectors e^ξ and e^{ξ^*} are zonoid equivalent if and only if*

$$\varphi_\xi(u - \imath w) = \varphi_{\xi^*}(u - \imath w) \tag{7.1}$$

for all $u \in \mathbb{R}^d$ with $\sum u_i = 0$ and for at least one (and then necessarily for all) w, such that $\sum w_k = 1$ and both sides in (7.1) are finite.

Assume that e^ξ and e^{ξ^*} are two random vectors, where ξ and ξ^* are infinitely divisible random variables. Then

$$\varphi_\xi(u) = \mathbf{E}e^{\imath\langle u,\xi\rangle} = \exp\left\{\imath\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d}(e^{\imath\langle u,x\rangle} - 1 - \imath\langle u, x\rangle\,\mathbb{1}_{\|x\|\le 1})d\nu(x)\right\},$$

for $u \in \mathbb{R}^d$, where $A = (a_{ij})$ is a symmetric non-negative definite $d \times d$ matrix, $b \in \mathbb{R}^d$ is a constant vector and ν is a measure on $\mathbb{R}^d$ (called the Lévy measure) satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d}\min(\|x\|^2, 1)d\nu(x) < \infty\,.$$

Then ξ is said to have the Lévy triplet (A, ν, b). In this section we translate the equality of the zonoids of two log-infinitely divisible random vectors into conditions on their Lévy triplets. Note that the conditions on the Lévy triplet of infinitely divisible random vectors apply also for Lévy processes with time one values ξ and ξ^*.

In order to formulate the condition on the Gaussian terms in a compact form it is helpful to use the *variogram*

$$\gamma_{ij} = a_{ii} + a_{jj} - 2a_{ij} .$$

If ξ is normally distributed, then γ_{ij} is the variance of $\xi_i - \xi_j$. In order to state the condition on the Lévy measure define $(d-1) \times d$-dimensional matrix, $d \geq 2$

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} .$$

Theorem 7.9. *Let e^ξ and e^{ξ^*} be integrable random vectors such that ξ and ξ^* are infinitely divisible with characteristic triplets (A, ν, γ) and (A^*, ν^*, γ^*). Then for $d \geq 2$ e^ξ and e^{ξ^*} are zonoid equivalent if and only if the following three conditions hold.*

(a) $\gamma_{ij} = \gamma^*_{ij}$ for all $i, j \in \{1, \dots, d\}$.

(b) The images $\hat{\nu}U^{-1}$ and $\hat{\nu}^*U^{-1}$ under U of measures $d\hat{\nu}(x) = e^{x_d}d\nu(x)$ and $d\hat{\nu}^*(x) = e^{x_d}d\nu^*(x)$, $x \in \mathbb{R}^d$, restricted to $\mathbb{R}^{d-1} \setminus \{0\}$ coincide.

(c) $\mathbf{E}e^{\xi_i} = \mathbf{E}e^{\xi^*_i}$ for all $i = 1, \dots, d$, i.e.

$$b_i + \frac{1}{2}a_{ii} + \int_{\mathbb{R}^d} (e^{x_i} - 1 - x_i \, \mathbb{1}_{\|x\| \leq 1}) d\nu(x)$$
$$= b^*_i + \frac{1}{2}a^*_{ii} + \int_{\mathbb{R}^d} (e^{x_i} - 1 - x_i \, \mathbb{1}_{\|x\| \leq 1}) d\nu^*(x) . \quad (7.2)$$

For $d = 1$, e^ξ and e^{ξ^*} are zonoid equivalent if and only if (c) holds.

Corollary 7.10. *Two lognormal random vectors e^ξ and e^{ξ^*} are zonoid equivalent if and only if $\mu_i + \frac{1}{2}a_{ii} = \mu^*_i + \frac{1}{2}a^*_{ii}$ for all i and $\gamma_{ij} = \gamma^*_{ij}$ for all i, j, i.e. ξ and ξ^* have identical variogram.*

In particular, in the lognormal case the zonoid equivalence does not even imply the equality of the marginal distributions, quite differently to the case of normal distributions where the zonoid uniquely determines the joint distribution, see Corollary 7.7.

Furthermore, note that the kernel of U is the family of vectors with all equal components. Hence, if the support of ν is a subset of the kernel of U, then the corresponding infinitely divisible distribution shares the same zonoid with a lognormal distribution, meaning that two rather different distributions are zonoid equivalent.

Proof of Theorem 7.9. For $d \geq 2$ the zonoid equivalence of e^{ξ} and e^{ξ^*} implies $\mathbf{E}e^{\xi} = \mathbf{E}e^{\xi^*}$, see Proposition 2.6, and in particular $c = \mathbf{E}e^{\xi_d} = \mathbf{E}e^{\xi_d^*}$. Note that this is also implied by *(c)*. Since also $Z_{e^\xi} = Z_{e^{\xi^*}}$ by Proposition 2.6,

$$\mathbf{E}(u_1 e^{\xi_1} + \cdots + u_d e^{\xi_d})_+ = \mathbf{E}e^{\xi_d}(u_1 e^{\xi_1 - \xi_d} + \cdots + u_{d-1} e^{\xi_{d-1} - \xi_1} + u_d)_+ \,,$$

the zonoid of e^{ξ} uniquely determines and is uniquely determined by the probability distribution of $U\xi = (\xi_1 - \xi_d, \ldots, \xi_{d-1} - \xi_d)$ under the probability measure $\mathbf{P}^d$ with density e^{ξ_d}/c.

In order to identify the distribution of $U\xi$ under $\mathbf{P}^d$ first note that the distribution of ξ under $\mathbf{P}^d$ has the characteristic triplet $(A, \hat{\nu}, \hat{b})$, where $d\hat{\nu}(x) = e^{x_d} d\nu(x)$ and

$$\hat{b} = b + \int_{\|x\| \leq 1} x(e^{x_d} - 1)\nu(dx) + Ae_d \,,$$

see [29, Ex. 7.3]. By [28, Prop. 11.10], the Lévy triplet of $U\xi$ under $\mathbf{P}^d$ is given by $A_U = UAU^\top$, $\hat{\nu}U^{-1}$ restricted onto $\mathbb{R}^{d-1} \setminus \{0\}$ and

$$b_U = U\hat{b} + \int_{\mathbb{R}^d} Ux(\mathbb{1}_{\|Ux\| \leq 1} - \mathbb{1}_{\|x\| \leq 1})\hat{\nu}(dx) \,.$$

The corresponding formula holds for ξ^*.

Equating the centred Gaussian terms, the Lévy measures, and simplifying $b_U = b_U^*$ yields that $U\xi$ under $\mathbf{P}^d$ coincides in distribution with $U\xi^*$ under $\mathbf{P}^{d*}$ if and only if

$$a_{ij} + a_{dd} - a_{di} - a_{jd} = a_{ij}^* + a_{dd}^* - a_{di}^* - a_{jd}^*, \qquad i, j = 1, \ldots, d-1, \tag{7.3}$$

condition *(b)* holds and, for all $i = 1, \ldots, d-1$,

$$\begin{aligned} b_i - b_d + a_{id} - a_{dd} + \int_{\mathbb{R}^d} (x_i - x_d)(\mathbb{1}_{\|Ux\| \leq 1}\, e^{x_d} - \mathbb{1}_{\|x\| \leq 1}) d\nu(x) \\ = b_i^* - b_d^* + a_{id}^* - a_{dd}^* + \int_{\mathbb{R}^d} (x_i - x_d)(\mathbb{1}_{\|Ux\| \leq 1}\, e^{x_d} - \mathbb{1}_{\|x\| \leq 1}) d\nu^*(x) \,. \end{aligned} \tag{7.4}$$

Adding equations (7.3) with $k, l = i, i$; $k, l = j, j$ (for given i and j), and subtracting (7.3) multiplied by two, we arrive at the equality of the variograms. Furthermore, noticing that

$$(a_{ij} + a_{dd} - a_{di} - a_{jd})_{ij=1}^{d-1} = \frac{1}{2}(\gamma_{id} + \gamma_{jd} - \gamma_{ij})_{ij=1}^{d-1}$$

we obtain that the equality of variograms implies (7.3). The equality of zonoids implies the equality of expectations, which exactly corresponds to (7.2). It remains to show that (7.2) together with other two conditions *(a)* and *(b)* imply (7.4).

By (7.2) we have for all $i = 1, \ldots, d-1$

$$b_i + \frac{1}{2}a_{ii} + \int_{\mathbb{R}^d} (e^{x_i} - 1 - x_i \, \mathbb{1}_{\|x\| \leq 1}) d\nu(x)$$
$$= b_i^* + \frac{1}{2}a_{ii}^* + \int_{\mathbb{R}^d} (e^{x_i} - 1 - x_i \, \mathbb{1}_{\|x\| \leq 1}) d\nu^*(x) , \quad (7.5)$$

$$b_d + \frac{1}{2}a_{dd} + \int_{\mathbb{R}^d} (e^{x_d} - 1 - x_d \, \mathbb{1}_{\|x\| \leq 1}) d\nu(x)$$
$$= b_d^* + \frac{1}{2}a_{dd}^* + \int_{\mathbb{R}^d} (e^{x_d} - 1 - x_d \, \mathbb{1}_{\|x\| \leq 1}) d\nu^*(x) , \quad (7.6)$$

while condition *(a)* implies

$$a_{ii} + a_{dd} - 2a_{id} = a_{ii}^* + a_{dd}^* - 2a_{id}^* \quad (7.7)$$

for all $i = 1, \ldots, d-1$. Furthermore, condition *(b)* implies

$$\int_{\mathbb{R}^d} (e^{x_i - x_d} - 1 - (x_i - x_d) \, \mathbb{1}_{\|Ux\| \leq 1}) d\hat{\nu}(x)$$
$$= \int_{\mathbb{R}^d} (e^{x_i - x_d} - 1 - (x_i - x_d) \, \mathbb{1}_{\|Ux\| \leq 1}) d\hat{\nu}^*(x) , \quad (7.8)$$

where $d\hat{\nu}(x) = e^{x_n} d\nu(x)$, since by changing variables

$$\int_{\mathbb{R}^{d-1}} (e^y - 1 - y \, \mathbb{1}_{\|y\| \leq 1}) d(\hat{\nu} U^{-1})(y) = \int_{\mathbb{R}^{d-1}} (e^y - 1 - y \, \mathbb{1}_{\|y\| \leq 1}) d(\hat{\nu}^* U^{-1})(y) .$$

Now (7.4) is obtained by subtracting from (7.5) the sum of (7.8), (7.6) and a half of (7.7).

Recall that equality of the zonoids is equivalent to equality of their support functions for all u on the unite sphere. Hence, for positive random variables e^{ξ} and e^{ξ^*} ($d = 1$) equality of their zonoids is equivalent to equality of their expectations, which in turn, is equivalent to condition (c). □

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